

### Solution 3

1. Let  $f$  be a function defined on  $(a, b)$  and  $x_0 \in (a, b)$ .

- (a) Show that  $f$  is Lipschitz continuous at  $x_0$  if its left and right derivatives exist at  $x_0$ .  
 (b) Construct a function Lipschitz continuous at  $x_0$  whose one sided derivatives do not exist.

**Solution.** (a) Let  $\alpha = f'_+(x_0)$ . For  $\varepsilon = 1 > 0$ , there exists  $\delta_1$  such that

$$\left| \frac{f(x+z) - f(x)}{z} - \alpha \right| < 1 ,$$

for  $0 < z < \delta_1$ . It follows that

$$|f(x+z) - f(x)| \leq |f(x+z) - f(x) - \alpha z| + |\alpha z| \leq (1 + |\alpha|)|z| .$$

Similarly,

$$|f(x+z) - f(x)| \leq (1 + |\alpha|)|z| , \quad z \in (-\delta_2, 0) .$$

We conclude that  $|f(x+z) - f(x)| \leq (1 + \delta)|z|$ ,  $z \in (-\delta, \delta)$ ,  $\delta = \min\{\delta_1, \delta_2\}$ .

(b) The function  $f(x) = x \sin \frac{1}{x}$  ( $x \neq 0$ ) and  $= 0$  at  $x = 0$ . It is Lipschitz continuous at  $x_0 = 0$  with  $L = 1$  but both one-sided derivatives do not exist.

2. Optional. Let  $f$  be a function defined on  $(a, b]$  which is integrable on  $[c, b]$  for all  $c \in (a, b)$ . It is called improperly integrable over  $(a, b]$  if

$$\lim_{c \rightarrow a^+} \int_c^b |f|$$

exists. When this happens,

$$\lim_{c \rightarrow a^+} \int_c^b f$$

also exists and we define the improper integral of  $f$  over  $(a, b]$  to be

$$\int_a^b f = \lim_{c \rightarrow a^+} \int_c^b f .$$

- (a) Show that if  $f$  is integrable on  $[a, b]$ , its improper integral also exists and is equal to its usual integral.  
 (b) Show that Riemann-Lebesgue Lemma holds for improperly integrable functions.

**Solution.** (a) Using

$$\int_a^c f \leq (c - a)M, \quad M = \sup |f| ,$$

for  $\varepsilon > 0$ ,

$$\left| \int_c^b |f| - \int_a^b |f| \right| = \int_a^c |f| \leq (c - a)M < \varepsilon ,$$

for all  $c, |c - a| < \varepsilon/M$ . Therefore,

$$\lim_{c \rightarrow a^+} \int_c^b |f| = \int_a^b |f| .$$

(b) For  $\varepsilon > 0$ , fix  $c \in (a, b)$  such that

$$\int_a^c |f| < \varepsilon/2 .$$

The function  $f$  is integrable on  $[c, b]$  and Riemann-Lebesgue Lemma applies to get

$$\left| \int_c^b f(x)e^{inx} dx \right| < \frac{\varepsilon}{2}, \quad n \geq n_0,$$

for some  $n_0$ . It follows that

$$\left| \int_a^b f(x)e^{inx} dx \right| \leq \int_a^c |f| + \left| \int_c^b f(x)e^{inx} dx \right| < \varepsilon, \quad \forall n \geq n_0 .$$

3. Optional. Show that

$$-\log \left| 2 \sin \frac{x}{2} \right| \sim \cos x + \frac{\cos 2x}{2} + \frac{\cos 3x}{3} + \dots .$$

Suggestion. I leave it to you to verify this function is  $2\pi$ -periodic and improperly integrable. The calculation of  $a_0$  is tricky, involving the definite integral  $I = \int_0^{\pi/2} \log \sin t dt$ . To evaluate it use  $\sin t = 2 \sin t/2 \cos t/2$  and eventually show  $I = -\frac{\pi}{2} \log 2$ .

**Solution.** I leave out the verification of periodicity and improper integrability. This is an even function, so its Fourier series is a cosine series. Let

$$\pi a_0 = \int_0^\pi \log \left( 2 \sin \frac{x}{2} \right) dx = \pi \log 2 + Y, \quad Y = \int_0^\pi \log \sin \frac{x}{2} dx.$$

We have

$$\begin{aligned} Y &= 2 \int_0^{\pi/2} \log \sin t dt \\ &= 2 \int_0^{\pi/2} \log \left( 2 \sin \frac{t}{2} \cos \frac{t}{2} \right) dt \\ &= \pi \log 2 + 2 \int_0^{\pi/2} \log \sin \frac{t}{2} dt + 2 \int_0^{\pi/2} \log \cos \frac{t}{2} dt \\ &= \pi \log 2 + 2 \int_0^{\pi/2} \log \sin \frac{t}{2} dt + 2 \int_{\pi/2}^\pi \log \sin \frac{t}{2} dt \\ &= \pi \log 2 + 2Y, \end{aligned}$$

so  $Y = -\pi \log 2$ . It follows that  $a_0 = 0$ . The calculations for  $a_n$  make use of by parts to get

$$a_n = \frac{1}{n\pi} \int_0^\pi \frac{\sin nx \cos(x/2)}{\sin(x/2)} dx$$

first, then by

$$\sin nx \cos \frac{x}{2} = \frac{1}{2} \left( \sin \left( n + \frac{1}{2} \right) x + \sin \left( n - \frac{1}{2} \right) x \right)$$

and finally use Property 3 of the Dirichlet kernel.

4. Let  $a_n, b_n$  be the Fourier coefficients of some  $f \in R_{2\pi}$ .

- (a) Show that for each  $r \in [0, 1)$ , the trigonometric series given by

$$a_0 + \sum_{k=1}^{\infty} r^k (a_k \cos kx + b_k \sin kx)$$

is uniformly convergent to some function in  $C_{2\pi}$ . Denote this function by  $f_r(x)$ .

- (b) Show that

$$f_r(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(z) f(x+z) dz,$$

where the **Poisson kernel**  $P_r$  is given by

$$P_r(z) = \frac{1-r^2}{1-2r \cos z + r^2}.$$

- (c) Let  $f$  be continuous at  $x$ . Show that  $\lim_{r \rightarrow 1} f_r(x) = f(x)$ .

The treatment is parallel to that for the Dirichlet kernel (the parameter  $n$  is now replaced by  $r$ ), but differs at the final step; we do not need Lipschitz continuity. Think about it.

**Solution.** Look up [SS]. We don't need Lipschitz continuity because Poisson kernel is positive, so the analog of Property IV of the Dirichlet kernel does not hold, which is good news.

5. (a) Can you find a cosine series which converges uniformly to the sine function on  $[0, \pi]$ ? If yes, find one.  
 (b) Is the series in (a) unique?  
 (c) Can you find a cosine series which converges pointwisely to the sine function on  $[-a, \pi]$  where  $a$  is a number in  $(0, \pi)$ ?

**Solution.** (a) Yes, extend the sine function on  $[0, \pi]$  to  $|\sin x|$ , an even,  $2\pi$ -periodic function. Since it is continuous, piecewise  $C^1$ , its cosine series converges uniformly to this extended function. In particular, this cosine series converges uniformly to  $\sin x$  on  $[0, \pi]$ .  
 (b) Yes, there is only one way to extend  $\sin x$  as an even function. (c) No, can't have even extension. (When a function is the pointwise limit of an even function, it must be even.)

6. Let  $f$  be an integrable function on  $[-\pi, \pi]$ . Show that for each  $\varepsilon > 0$ , there exists a trigonometric polynomial  $p$  satisfying  $p < f$  on  $[-\pi, \pi]$  and

$$\int_{-\pi}^{\pi} |f - p| < \varepsilon.$$

**Solution.** Given  $\varepsilon > 0$ , we can find a continuous function  $g + \varepsilon_1 < f$  satisfying

$$\int_a^b |f - g| < \frac{\varepsilon}{4}$$

for a small  $\varepsilon_1 > 0$ . ( $g$  comes from modifying a step function constructed using a Darboux lower sum.) Then we find a trigonometric polynomial  $q$  satisfying  $|g(x) - q(x)| < \min\{\varepsilon_1/2, \varepsilon/4(b-a)\}$ , so

$$\int_a^b |g - q| < \frac{\varepsilon}{4}.$$

The function  $p = q + \min\{\varepsilon_1/2, \varepsilon/(4(b-a))\}$  satisfies our requirement.

Note. Weierstrass Theorem asserts every continuous function can be approximated by polynomials in  $[a, b]$ . Here it is shown that every integrable function can be approximated by polynomials in integral sense (that is, in average sense).

7. Show that there is a countable subset of  $C[a, b]$  such that for each  $f \in C[a, b]$ , there is some  $\varepsilon > 0$  such that  $\|f - g\|_\infty < \varepsilon$  for some  $g$  in this set. Suggestion: Take this set to be the collection of all polynomials whose coefficients are rational numbers.

**Solution.** Let  $P_N$  the collection of all polynomials of the form  $p(x) = a_0 + a_1x + \cdots + a_Nx^n$  where  $a_j, j = 0, \dots, N$  are rational numbers. The map  $p \mapsto (a_0, a_1, \dots, a_N)$  is a one-to-one correspondence between  $P_N$  and  $\mathbb{Q}^N$ , which is countable. As the countable union of countable sets is again countable,  $P = \cup_{N=0}^\infty P_N$  is also countable. Now, by Weierstrass theorem, for each  $f \in [a, b]$  and  $\varepsilon > 0$ , there exists a polynomial  $q$  (with real coefficients) such that  $\|f - q\|_\infty < \varepsilon/2$ . We may approximate  $q$  by a polynomial  $p$  from  $P$  such that  $\|q - p\|_\infty < \varepsilon/2$ . It follows that  $\|f - p\|_\infty \leq \|f - q\|_\infty + \|q - p\|_\infty < \varepsilon$ .

8. Let  $f$  be continuous on  $[a, b] \times [c, d]$ . Show that for each  $\varepsilon > 0$ , there exists a polynomial  $p = p(x, y)$  so that

$$\|f - p\|_\infty < \varepsilon, \quad \text{in } [a, b] \times [c, d].$$

In fact, this result holds in arbitrary dimension.

**Solution.** Just like we approximate a continuous function by continuous piecewise linear function in the one dimensional case, WLOG we may assume  $f(x, y)$  is doubly  $2\pi$ -periodic, uniformly Lipschitz continuous in  $[-\pi, \pi]^2$ . For every  $\varepsilon > 0$ , we can find an  $N$  such that for all  $y$

$$|f(x, y) - \sum_{k=1}^N (a_k(y) \cos kx + b_k(y) \sin kx)| < \varepsilon, \quad \forall x. \quad (1)$$

(See the last paragraph.) From the expression

$$a_k(y) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x, y) \cos kx dx$$

we see that each  $a_k$  is in  $C_{2\pi}$ . By Weierstrass Approximation Theorem, we can find polynomials  $p_k(x)$  such that  $|a_k(x) - p_k(x)| < \varepsilon/N$  for all  $x$ , and  $q_k$  for  $b_k$  similarly. It follows that

$$\begin{aligned} & |f(x, y) - \sum_{k=1}^N (p_k(x) \cos kx + q_k(x) \sin kx)| \\ & \leq \left| f - \sum_{k=1}^N (a_k(x) \cos kx + b_k(x) \sin kx) \right| \\ & \quad + \left| \sum_{k=1}^N (a_k(x) \cos kx + b_k(x) \sin kx) - \sum_{k=1}^N (p_k(x) \cos kx + q_k(x) \sin kx) \right| \\ & < \varepsilon + 2N \times \frac{\varepsilon}{N} \\ & = 3\varepsilon. \end{aligned}$$

Finally, we approximate  $\cos kx$  and  $\sin kx$  by polynomials in  $x$  to complete the job.

We justify (1) by a compactness argument. For each fixed  $y$ , the function  $x \mapsto f(x, y)$  is uniformly Lipschitz continuous. For  $\varepsilon > 0$ , there exists some natural number  $N_y$  depending on  $y$  such that

$$|f(x, y) - \sum_{k=1}^{N_y} (a_k(y) \cos kx + b_k(y) \sin kx)| < \varepsilon, \quad \forall x.$$

By continuity, there is some interval  $(y - \delta_y, y + \delta_y)$  so that

$$|f(x, z) - \sum_{k=1}^{N_y} (a_k(z) \cos kx + b_k(z) \sin kx)| < \varepsilon, \quad \forall x, \text{ and } z \in (y - \delta_y, y + \delta_y).$$

All these intervals  $\{(y - \delta_y, y + \delta_y)\}, y \in [-\pi, \pi]$ , form an open covering of  $[-\pi, \pi]$ . By Open Covering Theorem (I taught it in MATH2050 many years ago. Supposedly it is still covered in this course. I will teach this course again in the coming semester. Definitely I will teach it), we can find a finite subcover  $\{(y_k - \delta_{y_k}, y_k + \delta_{y_k})\}, k = 1, \dots, M$ . Then (1) holds by taking  $N = \max\{N_{y_1}, \dots, N_{y_M}\}$ .